3. fundamental Theorem of Algebra :-

Statement :- Every Polynomial $f(x) \in C[x]$ factors into linear factors in $C[x]$

or

Every Polynomial $f(x) \in C[x]$ has Elements in C.

Proof :- Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Step I

$f(x) \in C[x]$ $a_i \in C$, $0 \le i \le n$.

$\overline{f(x)} = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n$

Let $g(x) = (x^2+1)\, \overline{f(x)}\, f(x)$

$\bar{g}(x) = (x^2+1)\, f(x)\, \overline{f(x)}$

$\bar{g}(x) = g(x)$

Hence $g(x) \in R[x]$

Let K be splitting field of $g(x)$ over R

s.t. $\frac{K}{R}$ is finite & normal

and $ch(R) = 0$

Then $\frac{K}{R}$ is separable.

$\therefore \frac{K}{R}$ is a Galois groups.

In galois Theory $O(G(\frac{K}{R})) \le [K:R]$

$|G(K/R)| = [K:R] = I$

Let $I = 2^m r$

II) We claim $r = 1$

By Sylow first Thm: $G = G(\frac{K}{R})$ has Sylow 2-Sub group. say H

$o(H) = p^m$, $o(H) = 2^m$

Let L be fixed field of H s.t. $L = K_H$

Acc to galois Theory

$H = G(K:L)$

$o(H) = o(G(K:L)) = [K:L]$ ∴ $[K:L] = 2^m$

Let $[K:R] = [K:L][L:R]$

$2^m r = [K:L][L:R]$

$2^m r = 2^m [L:R]$

∴ $[L:R] = r$

K
L
R

Now $[K:R] = [K:L][L:R]$

L be finite separable Extension of R hence it is simple.

Let $\alpha \in L$ and $h(x) \in R(X)$

$R(\alpha) = L$ s.t. $[L:R] = r$

$[R(\alpha):R] = r$ But $r = 1$ prime order

$\exists \beta$, $(x-\beta)$ which is factor of $h(x)$

Contradiction ∵ Given Polynomial is irreducible.

Step III:

Now $(x^2+1) / g(x)$ & C be the splitting field s.t. $C \subseteq K$

$$[K:R] = [K:C][C:R]$$

Let $[K:C] = 2^{m-1}$

We claim K has no sub field s.t.

$C \subseteq K_1$ s.t. $[K_1 : C] = 2$

& $[K:C] = 2$, Prime

Then K/C is normal.

$Ch(C) = 0$, K/C is separable.

Let $K_1 = C(\gamma)$

Degree $\gamma = [K_1 : C] = [C(\gamma):C] = 2$

Let $\phi(x) = x^2 + ax + b \in C[x]$

$$= x^2 + ax + \frac{a^2}{4} - \frac{a^2}{4} + b \in C[x]$$

$$= \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4} + b$$

$$= \left(x + \frac{a}{2}\right)^2 - \frac{a^2 - 4b}{4}$$

Let $\frac{a^2 - 4b}{4} = c^2$

$$\phi(x) = \left(x + \frac{a}{2}\right)^2 - c^2$$

$$= \left(x + \frac{a}{2} + c\right)\left(x + \frac{a}{2} - c\right)$$

Contradiction as $\phi(x)$ is irreducible.

Step IV We have $[K:C] = 2^{m-1}$

Let $m-1 \geq 0$

Let $m-1 = k$

$[K:C] = 2^k$

$O(G(K/C)) = [K:C] = 2^k$

$\exists$ a subgroup H_1

s.t. $O(H_1) = 2^{k-1}$

& K_1 be fixed field of H_1

Then $|H_1| = |G(K/K_1)| = [K:K_1] = 2^{k-1}$

Now $2^k = [K:C]$
$= [K:K_1][K_1:C]$

$2^k = 2^{k-1}[K_1:C]$

$2^k \cdot 2^{-k+1} = [K_1:C]$

$[K_1:C] = 2$ Prime No.

Contradiction
$k = m-1 = 0$
$\Rightarrow m = 1$

$[K:C] = 2^{1-1} = 2^0 = 1$

$[K:C] = 1$
i.e. $K = C$

Hence Proved.

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